Trigonometry Workbook Solutions

Six circular functions

krista king
MATH

DOMAIN AND RANGE OF THE SIX CIRCULAR FUNCTIONS

■ 1. Find the domain of $f(x)$.

$$f(x) = \sqrt{\sin x}$$

Solution:

Since the domain of the sine function is the set of all real numbers, we only need to find the domain for the square root function. Since the radicand must be nonnegative, we can say

$$\sin x \geq 0$$

We know that the sine function is equal to 0 at angles of 0 and π, and is positive between those. So, for the interval $[0,2\pi]$, the solution is

$$0 \leq x \leq \pi \text{ or } x \in [0,\pi]$$

Considering the sine period of 2π, we get

$$x \in \ldots \cup [-2\pi, -\pi] \cup [0,\pi] \cup [2\pi,3\pi] \cup [4\pi,5\pi] \cup \ldots$$

So we can write the solution as $x \in [2\pi n, \pi + 2\pi n]$, where n is an integer.

■ 2. Find the domain of $f(x)$.

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$$f(x) = \frac{2\tan x - 3\cot x}{2 - \sin 3x}$$

Solution:

Since the function consists of several circular functions, and each of them has its own domain, let's find the domain of each function individually, and then use them to find the domain of $f(x)$.

(1) Since the denominator of the fraction can't be 0,

$$2 - \sin 3x \neq 0$$

$$\sin 3x \neq 2$$

Since $\sin\theta \leq 1$, the equation $\sin 3x = 2$ has no solution, so the domain of the denominator is the set of all real numbers.

(2) The tangent function is undefined at

$$x = \frac{\pi}{2} + \pi n$$

where n is any integer. So the domain excludes

$$x = \pm\left(\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots\right)$$

(3) The cotangent function is undefined at

$$x = \pi m$$

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where m is any integer.

So the domain excludes

$$x = \pm\,(0,\ \pi,\ 2\pi,\ 3\pi,\ \dots)$$

Putting these exclusions together gives

$$x = \pm\left(0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi,\ \frac{5\pi}{2},\ 3\pi,\ \frac{7\pi}{2},\ 4\pi,\ \dots\right)$$

$$x = \pm\left(\frac{0\pi}{2},\ \frac{\pi}{2},\ \frac{2\pi}{2},\ \frac{3\pi}{2},\ \frac{4\pi}{2},\ \frac{5\pi}{2},\ \frac{6\pi}{2},\ \frac{7\pi}{2},\ \frac{8\pi}{2},\ \dots\right)$$

$$x = \pm\frac{\pi}{2}(0,\ 1,\ 2,\ 3,\ 4,\ \dots)$$

$x = \pm\frac{\pi}{2}n$, where n is any integer

So the domain of $f(x)$ is all real numbers, except

$x = \pm\frac{\pi}{2}n$, where n is any integer

3. Find the range of $f(x)$.

$$f(x) = 3\sec 2x - 1$$

Solution:

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We know that the range of the secant function is $(-\infty, -1] \cup [1,\infty)$. So let's transform the secant function until we get to $f(x)$, and we'll adjust the range every time we make a transformation.

First, compress the function by two times in the x-direction.

$$f_1(x) = \sec 2x$$

Since compressing or stretching in the x-direction doesn't change the range, the range is still $(-\infty, -1] \cup [1,\infty)$.

Second, stretch the function by three times in the y-direction.

$$f_2(x) = 3 \sec 2x$$

Stretching the function in the y-direction does change the range. So the range of $f_2(x)$ is

$$(-\infty, -1 \cdot 3] \cup [1 \cdot 3,\infty)$$

$$(-\infty, -3] \cup [3,\infty)$$

Third, shift the function down by 1 in the y-direction.

$$f_3(x) = 3 \sec 2x - 1$$

Shifting the function in the y-direction does change the range. So the range of $f_3(x)$ is

$$(-\infty, -3 - 1] \cup [3 - 1,\infty)$$

$$(-\infty, -4] \cup [2,\infty)$$

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Therefore, the range of $f(x)$ is $(-\infty, -4] \cup [2, \infty)$.

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FIND THE VALUE OR SAY WHERE IT'S UNDEFINED

1. Evaluate the expression.

$$2\csc\frac{49\pi}{6}-3\cos\frac{13\pi}{3}+\tan\frac{25\pi}{4}$$

Solution:

Reduce each angle to an angle that's coterminal, but in the interval $[0,2\pi)$.

So

$$\frac{49\pi}{6}=\frac{48\pi}{6}+\frac{\pi}{6}=8\pi+\frac{\pi}{6}$$

$$\frac{13\pi}{3}=\frac{12\pi}{3}+\frac{\pi}{3}=4\pi+\frac{\pi}{3}$$

$$\frac{25\pi}{4}=\frac{24\pi}{4}+\frac{\pi}{4}=6\pi+\frac{\pi}{4}$$

Which means

$$\csc\frac{49\pi}{6}=\csc\frac{\pi}{6}$$

$$\cos\frac{13\pi}{3}=\cos\frac{\pi}{3}$$

$$\tan\frac{25\pi}{4}=\tan\frac{\pi}{4}$$

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Then rewrite the expression as

$$2\csc\frac{\pi}{6}-3\cos\frac{\pi}{3}+\tan\frac{\pi}{4}$$

$$\frac{2}{\sin\frac{\pi}{6}}-3\cos\frac{\pi}{3}+\frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}}$$

Plug in the known values for the circular functions.

$$\frac{2}{\frac{1}{2}}-3\cdot\frac{1}{2}+\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$

$$4-\frac{3}{2}+1$$

$$\frac{7}{2}$$

■ 2. Find the angle $\theta \in [0,2\pi)$.

$$\sin\theta=\frac{1}{2} \text{ and } \cos\theta=-\frac{\sqrt{3}}{2}$$

Solution:

From the unit circle, we know that within the interval $[0,2\pi)$ the sine function takes the value of $1/2$ when $\pi/6$ and $5\pi/6$.

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So plug these values into the cosine equation. If $\theta = \pi/6$, then

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

This isn't the value of cosine that we're looking for, so $\theta = 5\pi/6$, but let's still double-check it.

$$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$

This is the value of cosine we were looking for, so $\theta = 5\pi/6$.

■ 3. Find the angle θ in the fourth quadrant.

$$\csc\theta = -2$$

Solution:

By the reciprocal identity,

$$\csc\theta = \frac{1}{\sin\theta}$$

If we substitute $\csc\theta = -2$,

$$-2 = \frac{1}{\sin\theta}$$

$$\sin\theta = -\frac{1}{2}$$

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From the unit circle, we know that the sine function takes negative values in the third and fourth quadrants, and that the value of $-1/2$ corresponds to the angles $7\pi/6$ and $11\pi/6$.

Since $7\pi/6$ doesn't lie in the fourth quadrant, only $11\pi/6$ satisfies the equation.

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FIND ALL SIX TRIG FUNCTIONS GIVEN THE QUADRANT OF THE ANGLE

1. Evaluate the expression if $\cos\theta = 1/\sqrt{3}$.

$$\tan^2\theta + \sin^2\theta + \sec^2\theta$$

Solution:

Since we don't know the exact value of θ, we need to use other ways to find the values of the circular functions.

Because there are only squares in the expression, we don't need to worry about the sign of the values, and we can just assume that each term is positive.

If we use the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, then we get

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\sin^2\theta = 1 - \left(\frac{1}{\sqrt{3}}\right)^2$$

$$\sin^2\theta = 1 - \frac{1}{3}$$

$$\sin^2\theta = \frac{2}{3}$$

If we use the quotient identity $\tan\theta = \sin\theta/\cos\theta$, then we get

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$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$$

$$\tan^2\theta = \frac{\frac{2}{3}}{\frac{1}{3}}$$

$$\tan^2\theta = 2$$

If we use the reciprocal identity $\sec\theta = 1/\cos\theta$, then we get

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\sec^2\theta = \frac{1}{\cos^2\theta}$$

$$\sec^2\theta = \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$\sec^2\theta = 3$$

Then the value of the expression is

$$\tan^2\theta + \sin^2\theta + \sec^2\theta$$

$$2 + \frac{2}{3} + 3$$

$$\frac{17}{3}$$

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■ 2. Find $\cot\theta$ if $\cos\theta = 0.6$ and $\theta \in [5\pi, 6\pi]$.

Solution:

We're told that θ is between 5π and 6π, so it lies in quadrant III or IV. We also know cosine is positive, and since cosine is only positive in quadrants I and IV, θ must lie in quadrant IV.

In quadrant IV, $\sin\theta$ is negative, which means cotangent is also negative. So we'll find sine's value first.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\sin^2\theta = 1 - 0.6^2$$

$$\sin^2\theta = 0.64$$

And then because we know sine is negative,

$$\sin\theta = -\sqrt{0.64}$$

$$\sin\theta = -0.8$$

Then $\cot\theta$ must be

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

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$$\cot\theta = \frac{0.6}{-0.8}$$

$$\cot\theta = -\frac{3}{4}$$

■ 3. Find $\sin\theta$ if $\sec\theta = 3$ and $\cot\theta < 0$.

Solution:

Use the reciprocal identity to find $\cos\theta$.

$$\sec\theta = \frac{1}{\cos\theta}$$

$$3 = \frac{1}{\cos\theta}$$

$$\cos\theta = \frac{1}{3}$$

The quotient identity tells us that

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Since we were told that cotangent is negative, and we just found that cosine is positive, sine must be negative. Then we can find $\sin\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

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$$\sin^2\theta = 1 - \cos^2\theta$$

$$\sin^2\theta = 1 - \frac{1}{9}$$

$$\sin^2\theta = \frac{8}{9}$$

And because we already know $\sin\theta$ must be negative,

$$\sin\theta = -\sqrt{\frac{8}{9}}$$

$$\sin\theta = -\frac{2\sqrt{2}}{3}$$

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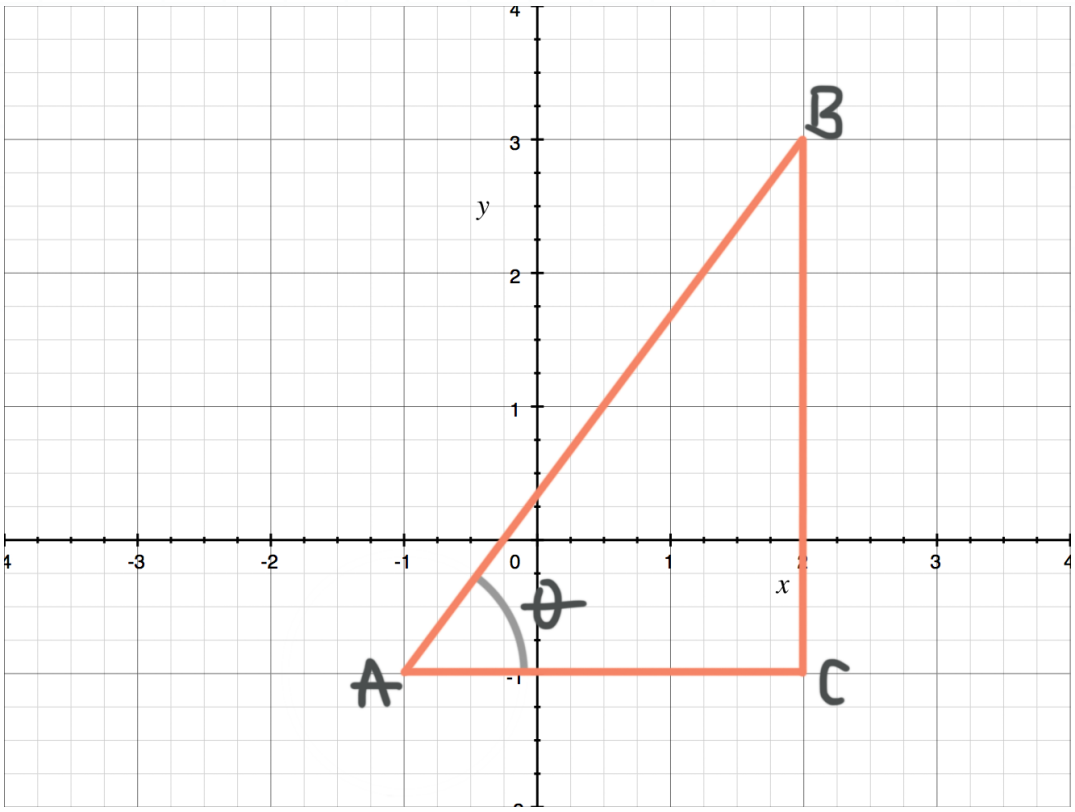
ANGLE OF ELEVATION

■ 1. Find $\cos\theta$, where θ is the angle of elevation of the point $(2,3)$ with respect to the point $(-1,-1)$.

Solution:

Consider the right triangle ABC such that $A(-1,-1)$, $B(2,3)$, and $C(2,-1)$. Then we'll say $\theta = \angle CAB$.



In triangle ABC, the length of AC is 3 and the length of BC is 4. In order to find the length of AB use the Pythagorean Theorem.

$$AC^2 + BC^2 = AB^2$$

$$3^2 + 4^2 = AB^2$$

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$$9 + 16 = AB^2$$

$$25 = AB^2$$

$$AB = 5$$

To find $\cos\theta$, use the definition of cosine, where cosine is equal to the length of the adjacent side divided by the length of the hypotenuse.

$$\cos\theta = \frac{AC}{AB} = \frac{3}{5}$$

■ 2. Chuck rides a ferris wheel at a carnival. Find the distance from Chuck to the ground (rounded to the nearest meter), if the center of the wheel is 18 m from the ground, the diameter of the wheel is 32 m, and that the angle of elevation of Chuck with respect to the center of the wheel is $60°$.

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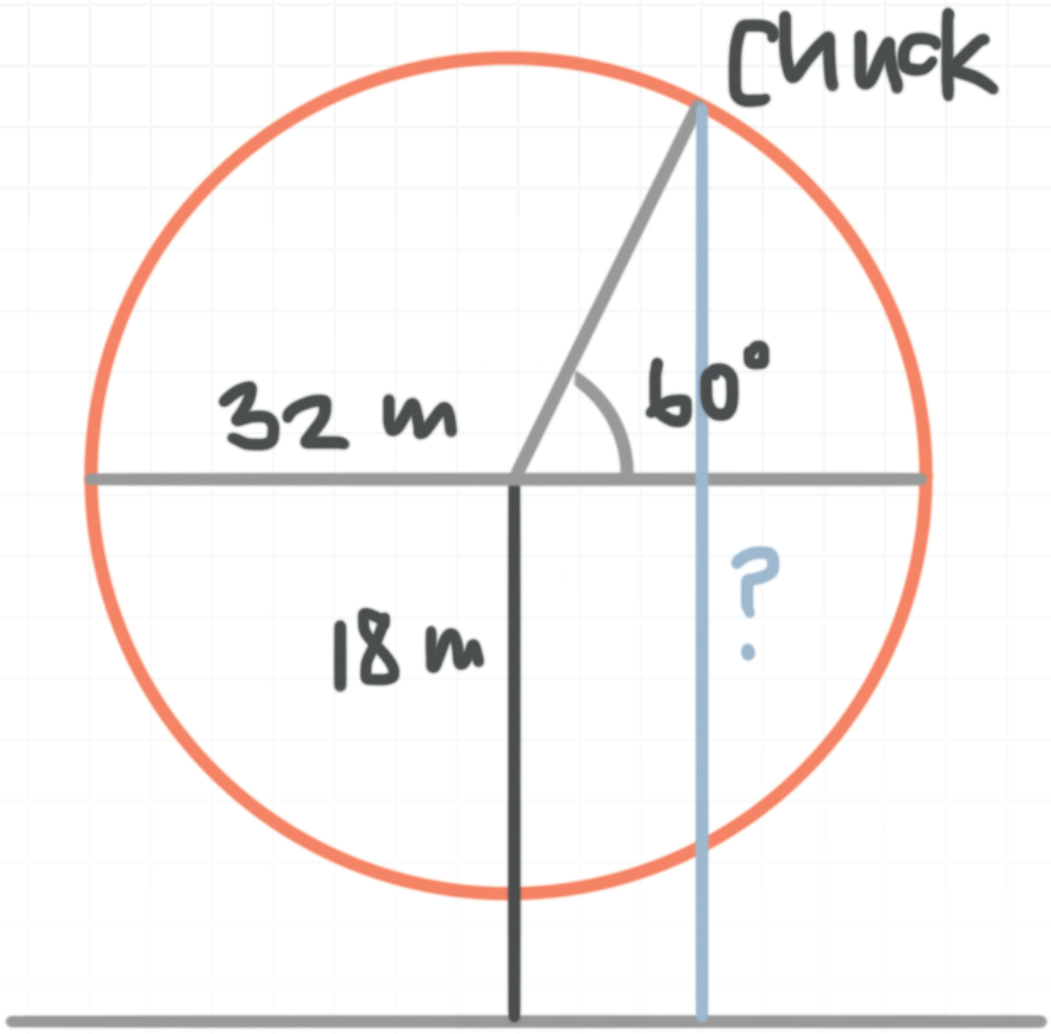

Solution:

Find the vertical distance from Chuck to the center of the wheel by creating a right triangle. If A is the center of the wheel and Chuck is at C, then

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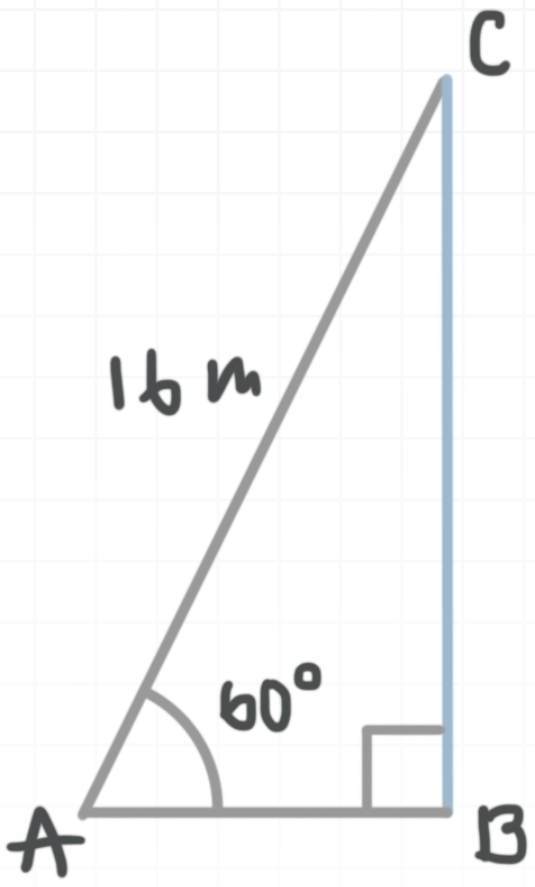

The angle $CAB = 60°$ is the angle of elevation, BC is the vertical distance we're interested in, and AC is the radius of the wheel. Since the radius is the half of the diameter,

$$AC = \frac{32}{2} = 16 \text{ m}$$

We can use the sine function to find BC, since the value of sine is equal to the length of the opposite side, divided by the length of the hypotenuse.

$$\sin CAB = \frac{BC}{AC}$$

$$BC = AC \sin CAB$$

$$BC = 16 \sin 60°$$

$$BC = 16\frac{\sqrt{3}}{2}$$

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$$BC = 8\sqrt{3}$$

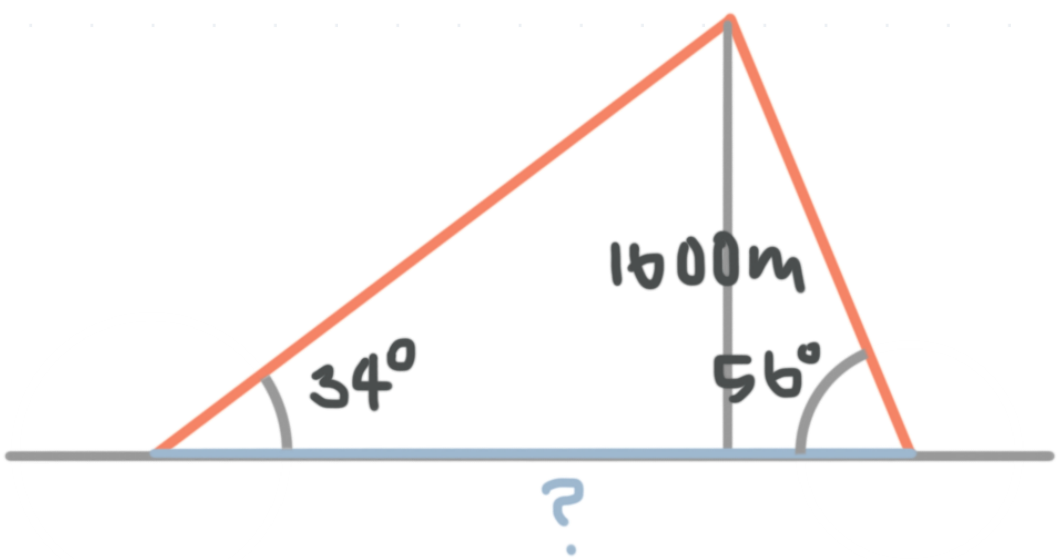
$$BC = \approx 14 \text{ m}$$

Since the vertical distance from Chuck to the center of the wheel is 14 m, and the distance from the center of the wheel to the ground is 18 m, then the distance from Chuck to the ground is

$$18 \text{ m} + 14 \text{ m}$$

$$32 \text{ m}$$

■ 3. A company considers building a tunnel through a mountain. The height of the mountain is $1{,}600$ m above the ground, and the angles of elevation of the peak with respect to the left and right endpoints of the tunnel are $34°$ and $56°$ respectively. Find the length of the tunnel to the nearest meter.



Solution:

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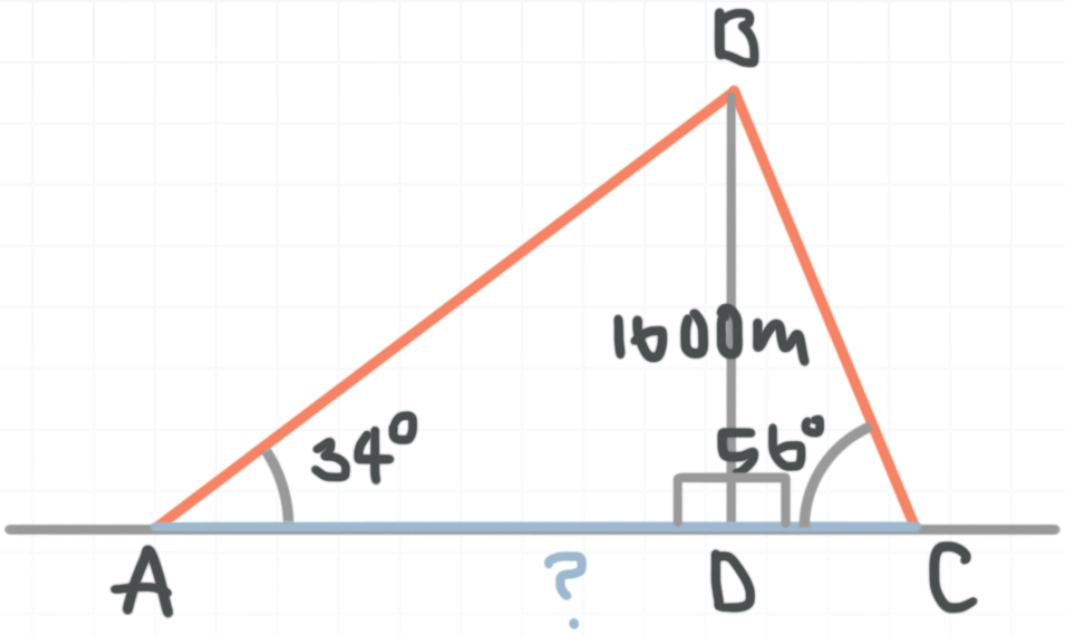
Let's set up the triangle with some extra information.



Now if we just look at triangle ADB, the angle of elevation BAD is $34°$, the length of DB is $1{,}600$ m, and the length of AD is the distance we need to find.

We can set up an equation using the tangent function, since tangent is equal to the length of the opposite side, divided by the length of the adjacent side.

$$\tan BAD = \frac{DB}{AD}$$

$$AD = \frac{DB}{\tan BAD} = \frac{1{,}600}{\tan 34°} \approx 2{,}372 \text{ m}$$

Now if we just look at triangle CDB, we can use a similar set of information to set up a tangent equation.

$$\tan BCD = \frac{DB}{CD}$$

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$$CD = \frac{DB}{\tan BCD} = \frac{1{,}600}{\tan 56^\circ} \approx 1{,}079 \text{ m}$$

Then the length of the tunnel is

$$AD + CD$$

$$2{,}372 \text{ m} + 1{,}079 \text{ m}$$

$$3{,}451 \text{ m}$$

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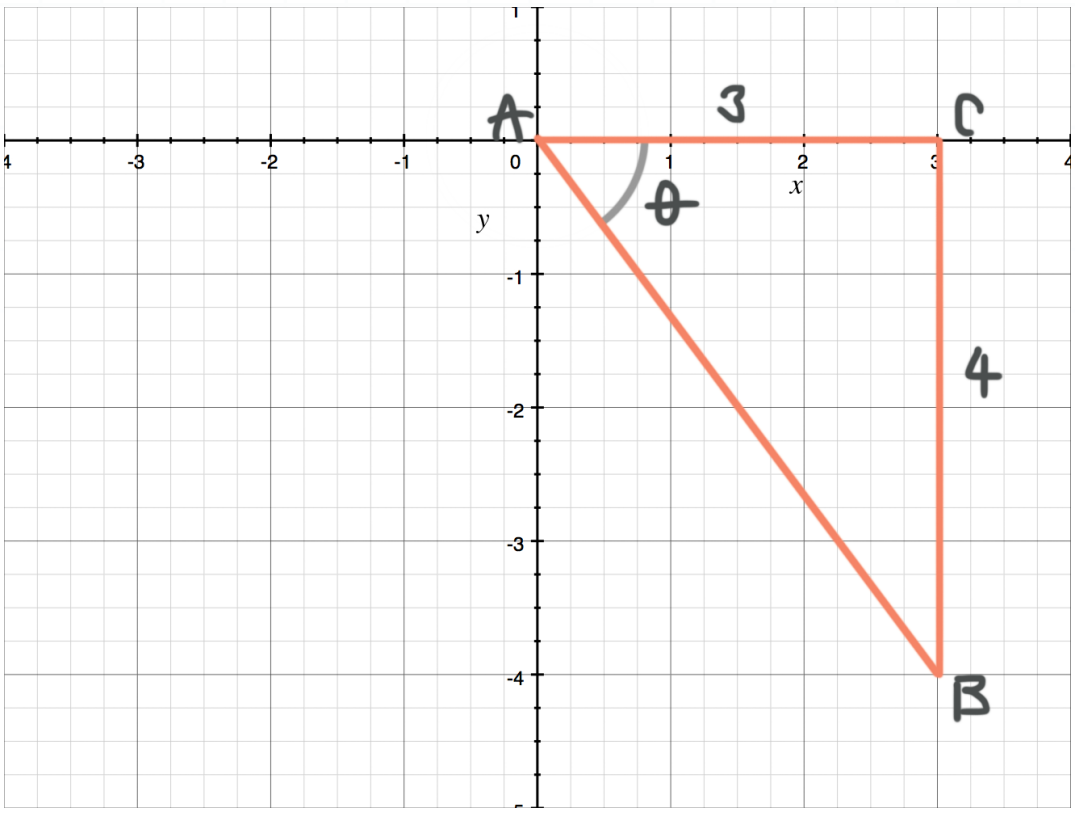
ANGLE OF DEPRESSION

■ 1. Find $\sin\theta$, where θ is the angle of depression of the point $(3, -4)$ with respect to origin.

Solution:

Make a right triangle from the two points, where A is at the origin, and B is at $(3, -4)$.



Use the Pythagorean Theorem to find the length of AB.

$$AC^2 + BC^2 = AB^2$$

$$3^2 + 4^2 = AB^2$$

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$$9 + 16 = AB^2$$

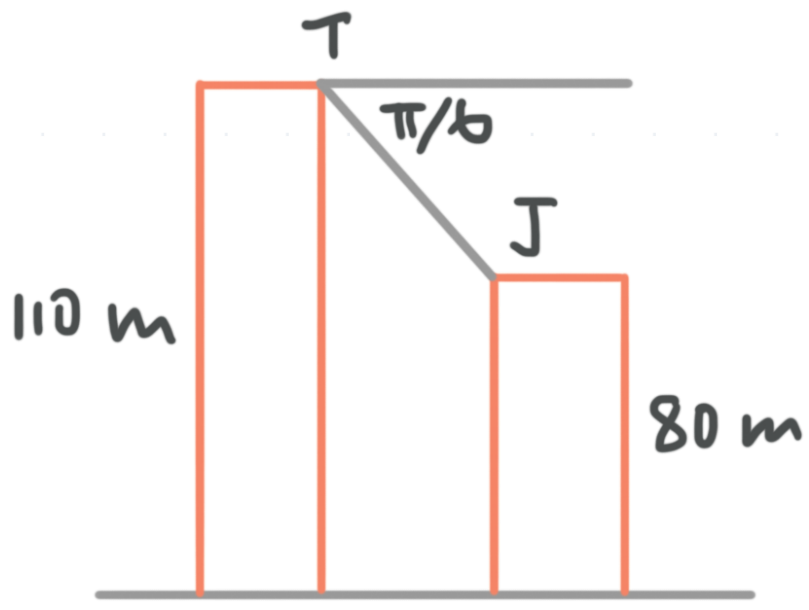
$$25 = AB^2$$

$$AB = 5$$

Then we can find $\sin\theta$ as the length of the opposite side, divided by the length of the hypotenuse.

$$\sin\theta = \frac{BC}{AC} = \frac{4}{5}$$

■ 2. Tom stands on the roof of a 110 m high building, and he looks down at Jerry, who's standing on the roof of a neighboring 80 m high building. The angle of depression of Jerry with respect to Tom is $\pi/6$ radians. Find the distance between the houses to the nearest meter.



Solution:

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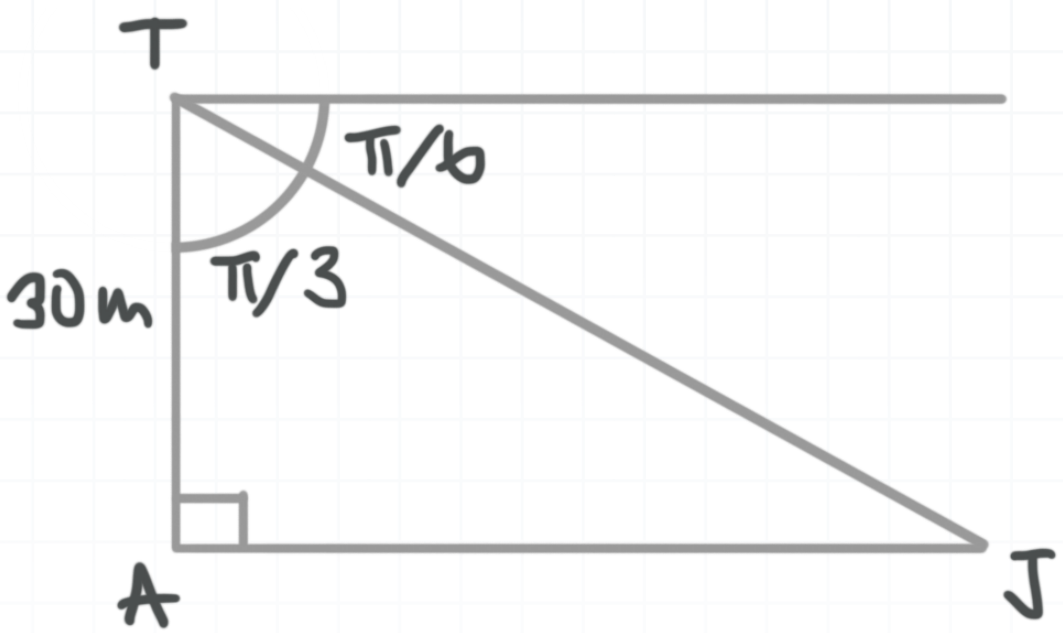
Sketch a triangle that includes the points where Tom and Jerry are standing.



In triangle TJA, angle ATJ is

$$\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3} \text{ radians}$$

The length of side AT is the difference between building heights.

$$AT = 110 - 80 = 30 \text{ m}$$

Then because tangent of an angle is equal to the length of the opposite side, divided by the length of the adjacent side, we can set up a tangent equation.

$$\tan ATJ = \frac{AJ}{AT}$$

$$AJ = AT \tan ATJ$$

$$AJ = 30 \tan \frac{\pi}{3} = 30\sqrt{3} \approx 52 \text{ m}$$

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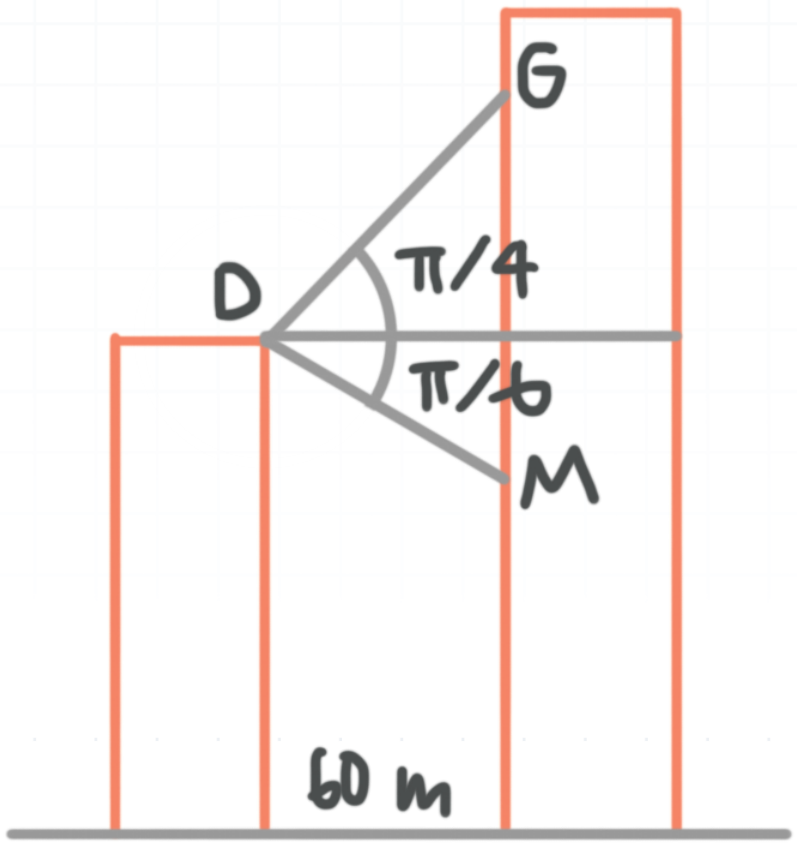
■ 3. Donald stands on the roof of a building, and he can see his friends Mickey and Goofy standing on different floors of a neighboring building. The angle of depression of Mickey with respect to Donald is $\pi/6$ radians. The angle of elevation of Goofy with respect to Donald is $\pi/4$ radians. Find the distance between Mickey and Goofy to the nearest tenth of a meter, if the distance between the buildings is 60 m.



Solution:

We can focus on the triangle created by Donald, Goofy, and Mickey.

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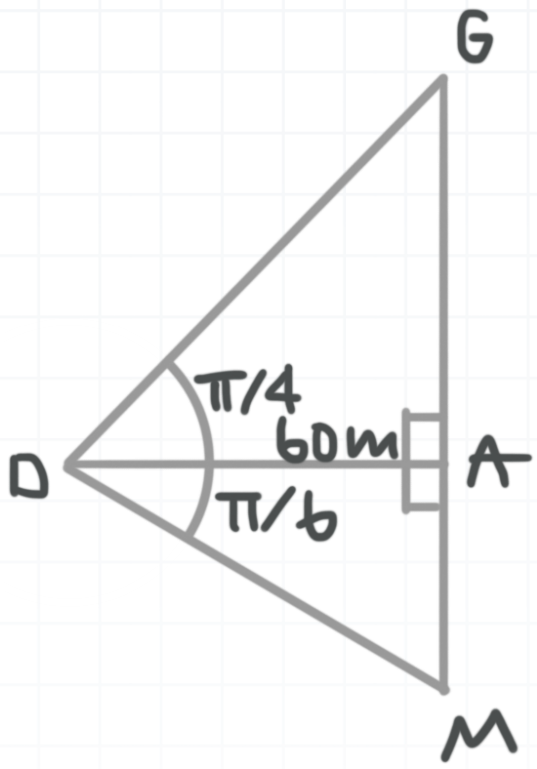

Looking just at triangle DAM, we can use the tangent function to write an equation.

$$\tan MDA = \frac{AM}{DA}$$

$$AM = DA \tan MDA$$

$$AM = 60 \tan \frac{\pi}{6} = 60\frac{\sqrt{3}}{3} = 20\sqrt{3} \approx 34.6 \text{ m}$$

Now look just at triangle DAG. We could use a tangent function just like we did with the other triangle, but the right triangle with an acute angle of $\pi/4$ is isosceles. So DA and AG are both 60 m.

So the distance between Mickey and Goofy is

$$AM + AG$$

$$60 \text{ m} + 34.6 \text{ m}$$

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94.6 m

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PROVING THE TRIG EQUATION

■ 1. Prove that the product of all six circular functions is equal to 1 for any angle θ lying within their domains.

$$\sin\theta \cdot \cos\theta \cdot \tan\theta \cdot \cot\theta \cdot \sec\theta \cdot \csc\theta = 1$$

Solution:

The reciprocal identity with sine and cosecant is

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sin\theta \cdot \csc\theta = 1$$

So we could simplify the equation to

$$1 \cdot \cos\theta \cdot \tan\theta \cdot \cot\theta \cdot \sec\theta = 1$$

$$\cos\theta \cdot \tan\theta \cdot \cot\theta \cdot \sec\theta = 1$$

The reciprocal identity with cosine and secant is

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cos\theta \cdot \sec\theta = 1$$

So we could further simplify the equation to

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$$1 \cdot \tan\theta \cdot \cot\theta = 1$$

$$\tan\theta \cdot \cot\theta = 1$$

Then if we divide both sides of the remaining equation by $\cot\theta$ we get

$$\tan\theta = \frac{1}{\cot\theta}$$

We know this is a true equation, because it's the reciprocal identity for tangent, so we've proven the original equation.

2. Prove the identity.

$$\cos^4\theta - \sin^4\theta = (\cos\theta + \sin\theta)(\cos\theta - \sin\theta)$$

Solution:

Rewrite the left side of the equation.

$$\cos^4\theta - \sin^4\theta = (\cos\theta + \sin\theta)(\cos\theta - \sin\theta)$$

$$(\cos^2\theta)^2 - (\sin^2\theta)^2 = (\cos\theta + \sin\theta)(\cos\theta - \sin\theta)$$

Then the left side is a difference of squares that can be factored as

$$a^2 - b^2 = (a + b)(a - b)$$

Factoring gives

$$(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) = (\cos\theta + \sin\theta)(\cos\theta - \sin\theta)$$

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By the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$,

$$1(\cos^2\theta - \sin^2\theta) = (\cos\theta + \sin\theta)(\cos\theta - \sin\theta)$$

$$\cos^2\theta - \sin^2\theta = (\cos\theta + \sin\theta)(\cos\theta - \sin\theta)$$

The left side is again the difference of squares.

$$(\cos\theta + \sin\theta)(\cos\theta - \sin\theta) = (\cos\theta + \sin\theta)(\cos\theta - \sin\theta)$$

Because the left and right sides of the equation are equivalent, we've proven the identity.

3. Prove the identity.

$$\frac{1 + \tan\theta}{1 + \cot\theta} = \tan\theta$$

Solution:

Use the reciprocal identity $\tan\theta = 1/\cot\theta$ to rewrite the right side.

$$\frac{1 + \tan\theta}{1 + \cot\theta} = \frac{1}{\cot\theta}$$

Create a common denominator on both sides of the equation.

$$\frac{1 + \tan\theta}{1 + \cot\theta} \cdot \frac{\cot\theta}{\cot\theta} = \frac{1}{\cot\theta} \cdot \frac{1 + \cot\theta}{1 + \cot\theta}$$

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$$\frac{\cot\theta(1+\tan\theta)}{\cot\theta(1+\cot\theta)} = \frac{1+\cot\theta}{\cot\theta(1+\cot\theta)}$$

In order for the equation to be true, the numerators must be equivalent.

$$\cot\theta(1+\tan\theta) = 1+\cot\theta$$

$$\cot\theta + \tan\theta\cot\theta = 1+\cot\theta$$

$$\tan\theta\cot\theta = 1$$

If we divide both sides by $\cot\theta$, we get

$$\tan\theta = \frac{1}{\cot\theta}$$

We know this is a true equation, because it's the reciprocal identity for tangent, so we've proven the original equation.

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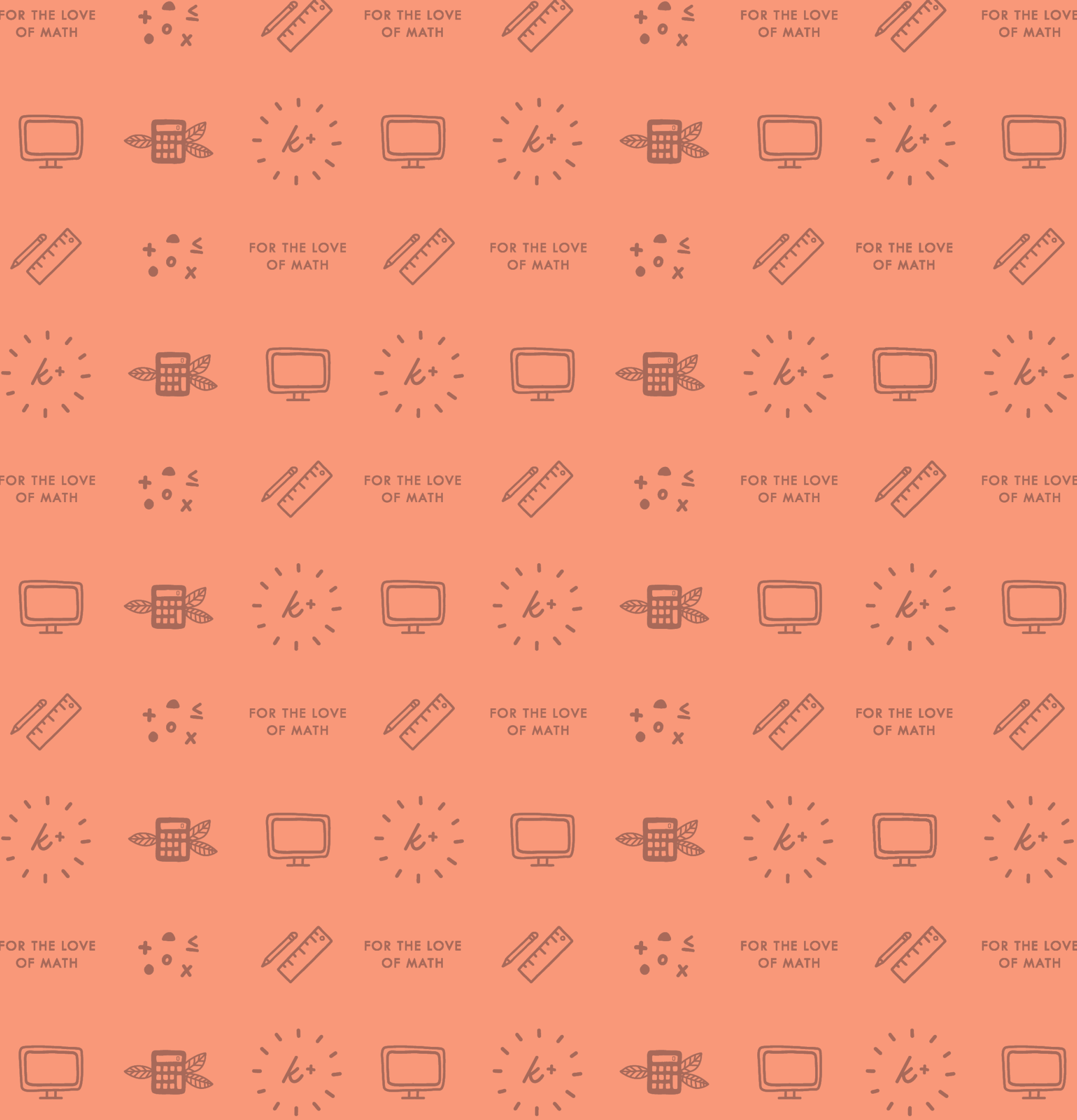
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